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Boolean D-Posets as the Factor Spaces

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In this paper the Boolean D-posets, as factor spaces of the Borel sets of the real line, are introduced.

1. INTRODUCTION

D-posets (Kôpka, 1992; Kôpka and Chovanec, 1994) (effect algebras; Foulis and Bennett, 1994) are the algebraic models of quantum mechanics. From this point of view the algebraic characteristics and some questions of probability theory on D-posets are studied in Kôpka (1995), Chovanec and Kôpka (n.d.), Greechie *et al.* (1995), Dvurečenskij and Pulmannová (1994), and Jurečková and Riečan (1995). In this paper a way of factorizing the system of Borel subsets of the real line is introduced which gives a Boolean D-poset. Thus, in the classical theory nonstandard access can be exploited for the solution of probability problems on D-posets.

Let $(P, \leq)$ be a nonempty partially ordered set (poset). A partial binary operation $\ominus$ is called a *difference* on P, and an element $b \ominus a$ is defined in P if and only if $a \leq b$ and the following conditions are satisfied:

(D1) $b \ominus a \leq b$.
(D2) $b \ominus (b \ominus a) = a$.
(D3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Let $(P, \leq, \ominus)$ be a poset with a difference and let 1 be the greatest element in P. The structure $(P, \leq, \ominus, 1)$ is called a *D-poset.* A D-poset $(P, \leq, \ominus, 1)$ satisfying the condition

(D4) if $(a_n)_{n=1}^{\infty} \subseteq P$, $a_n \leq a_{n+1}$ for any $n \in N$, then $\vee_{n=1}^{\infty} a_n \in P$

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is called a *D-σ-poset.* $(P, \vee, \wedge, \ominus, 1, 0)$ is called a D-lattice.

We say that the elements $a, b \in P$ are compatible, if $\exists d \in P, d \leq a, d \leq b$, and $a \ominus d \leq 1 \ominus b$ (equivalently, $b \ominus d \leq 1 \ominus a$).

Let P and T be two D-σ-posets. A mapping $w: P \to T$ is called a morphism of D-σ-posets if the following conditions are satisfied:

(M1) $w(1_P) = 1_T$.
(M2) If $(a_n)_{n=1}^{\infty} \subseteq P, a \in P, a_n \nearrow a$, then $w(a_n) \nearrow w(a)$.
(M3) If $a, b \in P, a \leq b$, then $w(b \ominus a) = w(b) \ominus w(a)$.

If P is the σ-algebra of Borel sets of the real line R, then the morphism $x: \mathcal{B}(R) \to T$ is called an observable on T. The spectrum of an observable $x: \mathcal{B}(R) \to T$ is the least closed subset F of $\mathcal{B}(\mathcal{R})$ such that $X(F) = 1$.

A poset $\mathcal{P}$ with the least element 0 and the greatest element 1 is said to be a *Boolean D-poset* if there is a binary operation "$-$" on $\mathcal{P}$ satisfying the following conditions:

(BD1) $a - 0 = a \; \forall \, a \in \mathcal{P}$.
(BD2) $a - (a - b) = b - (b - a) \; \forall a, b \in \mathcal{P}$.
(BD3) $a, b \in \mathcal{P}, a \leq b \Rightarrow c - b \leq c - a \; \forall \, c \in \mathcal{P}$.
(BD4) $(a - b) - c = (a - c) - b \; \forall a, b, c \in \mathcal{P}$.

2. BOOLEAN D-POSETS AS THE FACTOR SPACES

Let $\mathcal{B}(R)$ be the σ-algebra of all Borel sets of a real line. Let $A \in \mathcal{B}(R)$. We denote $\mathcal{B}_A(R) = \{E \in \mathcal{B}(R), E \subseteq A\}$.

Definition 1. Let $x: \mathcal{B}(R) \to P$ be an observable on D-posets. We say that the sets $A, B \in \mathcal{B}(R)$ are isomorphic by the observable x if there exists an isomorphism $\phi: \mathcal{B}_A(R) \to \mathcal{B}_B(R)$ such that $x(\phi(E)) = x(E)$ for every $E \in \mathcal{B}_A(R)$. We write $A \simeq_x B$. We remark that the relation $\simeq_x$ is an equivalence relation:

Definition 2. We say that the sets $A, B \in \mathcal{B}(R)$ are in the relation α_x (write $A \, \alpha_x \, B$), if there exist the sets $A_1, B_1 \in \mathcal{B}(R), A_1 \subseteq A, B_1 \subseteq B$ such that:
1. $x(A_1) = x(A), x(B_1) = x(B)$.
2. $A_1 \simeq_x B_1$.

Proposition 1. Let $A \, \alpha_x \, B$; then $x(A) = x(B)$.

Proof. Let $A_1 \subseteq A, B_1 \subseteq B$ such that the conditions 1 and 2 from Definition 2 are fulfilled. Let $\phi: \mathcal{B}_{A_1}(R) \to \mathcal{B}_{B_1}(R)$ be an isomorphism, such that $x(\phi(E)) = x(E)$ for every $E \in \mathcal{B}_{A_1}(R)$. Then

$$x(B) = x(B_1) = x(\phi(A_1)) = x(A_1) = x(A) \quad \blacksquare$$

Proposition 2. Let $A, B \in \mathcal{B}(R)$, $A \subseteq B$. Then $A\ \alpha_x\ B$ if and only if $x(A) = x(B)$.

Proof. The necessary condition is evident. Let $A \subseteq B$ and $x(A) = x(B)$. We put $A_1 = A$, $B_1 = A$. Then the mapping $\phi: \mathcal{B}_A(R) \to \mathcal{B}_A(R)$ such that $\phi(E) = E$ for every $E \in \mathcal{B}_A(R)$ is an isomorphism and $x(\phi(E)) = x(E)$ for every $E \in \mathcal{B}_A(R)$. Therefore $A\ \alpha_x\ B$. $\blacksquare$

Proposition 3. Let x: $\mathcal{B}(R) \in P$ be an observable on a D-poset P. Let $A_1, A_2, A \in \mathcal{B}(R)$, $A_1 \subseteq A$, $A_2 \subseteq A$. Now, $A_1\ \alpha_x\ A$ and $A_2\ \alpha_x\ A$ if and only if $(A_1 \cap A_2)\ \alpha_x\ A$.

Proof. Let $A_1 \subseteq A$, $A_2 \subseteq A$ and $A_1\ \alpha_x\ A$, $A_2\ \alpha_x\ A$. By Proposition 2 we have

$$x(A \backslash A_1) = x(A) \ominus x(A_1) = 0 = x(A) \ominus x(A_2) = x(A \backslash A_2)$$

Then

$$x(A_1 \cap A_2) = x(A_1 \backslash (A_1 \cap (A \backslash A_2))) = x(A_1) \ominus x(A_1 \cap (A \backslash A_2))$$
$$= x(A_1) \ominus 0 = x(A_1) = x(A)$$

By Proposition 2, $(A_1 \cap A_2)\ \alpha_x\ A$. The opposite assertion is evident. $\blacksquare$

Theorem 1. The relation α_x is an equivalence relation on $\mathcal{B}(R)$.

Proof. The reflexivity and symmetry are evident. We need to prove the transitivity of α_x. Let $A, B, C \in \mathcal{B}(R)$, $A\ \alpha_x\ B$, and $B\ \alpha_x\ C$, i.e., there exist the sets $A_1, B_1, B_2, C_2 \in \mathcal{B}(R)$, $A_1 \subseteq A$, $B_1 \subseteq B$, $B_2 \subseteq B$, $C_2 \subseteq C$ such that $x(A_1) = x(A)$, $x(B_1) = x(B) = x(B_2)$, $x(C_2) = x(C)$, and $A_1 \simeq_x B_1$, $B_2 \simeq_x C_2$.

Evidently $B_1\ \alpha_x\ B$, $B_2\ \alpha_x\ B$, which is equivalent to $(B_1 \cap B_2)\ \alpha_x\ B$. Let ϕ_1 and ϕ_2 be isomorphisms, ϕ_1: $\mathcal{B}_{A_1}(R) \to \mathcal{B}_{B_1}(R)$, ϕ_2: $\mathcal{B}_{B_2}(R) \to \mathcal{B}_{C_2}(R)$ such that $x(\phi_1(E)) = x(E)$ for every $E \in \mathcal{B}_{A_1}(R)$, $x(\phi_2(F)) = x(F)$ for every $F \in \mathcal{B}_{B_2}(R)$. We denote $A_0 = \phi_1^{-1}(B_1 \cap B_2)$. By the previous propositions we have

$$x(A_0) = x(\phi_1(\phi_1^{-1}(B_1 \cap B_2))) = x(B_1 \cap B_2) = x(B) = x(A)$$

Let $(\phi_1 \circ \phi_2)(A_0) = C_0$. Then

$$x(C_0) = x(\phi_2(\phi_1(A_0))) = x(\phi_1(A_0)) = x(A_0) = x(B) = x(C)$$

The mapping ψ: $\mathcal{B}_{A_0}(R) \to \mathcal{B}_{C_0}(R)$ defined by the formula $\psi(E) = (\phi_1 \circ \phi_2)(E)$ for every $E \in \mathcal{B}_{A_0}(R)$ is an isomorphism, and

$$x(\psi(E)) = x(\phi_2(\phi_1(E))) = x(\phi_1(E)) = x(E)$$

Therefore $A_0 \simeq_x C_0$. We have $A_0 \subseteq A$, $C_0 \subseteq C$, $x(A_0) = x(A)$, $x(C_0) = x(C)$, and $A_0 \simeq_x C_0$, which is equivalent to $A\ \alpha_x\ C$. $\blacksquare$

Corollary 1. Let $A, B, C \in \mathcal{B}(R)$, $A\ \alpha_x\ B$, $x(C) = 0$. Then $(A \cup C)\ \alpha_x\ B$, $A\ \alpha_x\ (B \cup C)$, $(A \cup C)\ \alpha_x\ (B \cup C)$.

Corollary 2. Let $A, B \in \mathcal{B}(R)$, $A\ \alpha_x\ B$, $A_1, B_1 \in \mathcal{B}(R)$, $A_1 \subseteq A$, $B_1 \subseteq B$, $x(A_1) = 0$, $x(B_1) = 0$. Then $(A \setminus A_1)\ \alpha_x\ (B \setminus B_1)$.

The factor space of $\mathcal{B}(R)$ corresponding to the equivalence relation α_x is denoted $\mathcal{B}(R)/\alpha_x = \{[A], A \in \mathcal{B}(R)\}$, where $[A] = \{E \in \mathcal{B}(R), E\ \alpha_x\ A\}$.

Definition 3. Let $[A], [B] \in \mathcal{B}(R)/\alpha_x$. We say that the element $[A]$ is less or equal to an element $[B]$ (denoted by $[A] \leq [B]$) if for every $A \in [A]$ there exists a Borel set $B \in [B]$ such that $A \subseteq B$.

It is evident that $[A] \leq [A]$ for every $[A] \in \mathcal{B}(R)/\alpha_x$. Let now $[A] \leq [B]$ and $[B] \leq [A]$. Then for every $A_1 \in [A]$ there exists $B \in [B]$ and $A_2 \in [A]$ such that $A_1 \subseteq B \subseteq A_2$. Since $x(A_1) = x(A_2)$, then $x(A_1) = x(B)$. By Proposition 2, $A_1\ \alpha_x\ B$, which is $A_1 \in [B]$. In an analogous way we prove that if $B_1 \in [B]$, then $B_1 \in [A]$. Therefore $[A] = [B]$.

Let $A, B, C \in \mathcal{B}(R)$ and $[A] \leq [B] \leq [C]$. Then for every $A \in [A]$ there exist $B \in [B]$ and $C \in [C]$ such that $A \subseteq B \subseteq C$. Therefore $[A] = [C]$.

Theorem 2. The relation $\leq$ on $\mathcal{B}(R)/\alpha_x$ from Definition 3 is a partial ordering on $\mathcal{B}(R)/\alpha_x$.

Proposition 4. Let $A, B \in \mathcal{B}(R)$, $A\ \alpha_x\ B$. Then $(A \setminus B)\ \alpha_x\ (B \setminus A)$.

Proof. Without loss of generality we may assume that there exists an isomorphism $\phi: \mathcal{B}_A(R) \to \mathcal{B}_B(R)$ and $x(E) = x(\phi(E))$ for every $E \in \mathcal{B}_A(R)$. We need to construct an isomorphism

$$\psi: \mathcal{B}_{A' \subseteq (A \setminus B)}(R) \to \mathcal{B}_{B' \subseteq (B \setminus A)}(R)$$

such that $x(E) = x(\psi(E))$ for every $E \in A'$. We denote $B_0 = B \setminus A$, $A_0 = A \setminus B$. Recursively we construct the following sequences of subsets of the sets A and B:

$$\begin{array}{lll}
B_1 = \phi(A_0) \cap B_0 & C_1 = \phi(A_0) \cap A \cap B & A_1 = \phi^{-1}(C_1) \\
B_2 = \phi^2(A_1) \cap B_0 & C_2 = \phi^2(A_1) \cap A \cap B & A_2 = \phi^{-2}(C_2) \\
B_3 = \phi^3(A_2) \cap B_0 & C_3 = \phi^3(A_2) \cap A \cap B & A_3 = \phi^{-3}(C_3) \\
\vdots & \vdots & \vdots \\
B_n = \phi^n(A_{n-1}) \cap B_0 & C_n = \phi^n(A_{n-1}) \cap A \cap B & A_n = \phi^{-n}(C_n) \\
\vdots & \vdots & \vdots
\end{array}$$

It is evident that $A_{i+1} \subseteq A_i\ \forall_i = 0, 1, \ldots$; $C_i \cap C_j = \emptyset\ \forall i \neq j$, $i, j = 1, 2,$

$\ldots$; $B_i \cap B_j = \emptyset \; \forall i \neq j,\ i, j = 1, 2, \ldots$; $\phi(C_i) = B_{i+1} \cup C_{i+1} \; \forall_i = 1, 2, \ldots$; $\phi^i(A_{i-1}\setminus A_i) = B_i \; \forall_i = 1, 2, \ldots$.

We denote $\cup_{i=1}^{\infty} C_i = C$, $(A \cap B)\setminus C = H$, $B_0\setminus(\cup_{i=1}^{\infty} B_i) = G$, $\cap_{i=1}^{\infty} A_i = E$. Then we have

$$\phi(A \cap B) = \phi(A\setminus A_0) = \phi(A)\setminus\phi(A_0) = B\setminus(C_1 \cup B_1)$$
$$= ((A \cap B)\setminus C_1) \cup (B_0\setminus B_1) = \left(\bigcup_{i=2}^{\infty} C_i\right) \cup \left(\bigcup_{i=2}^{\infty} B_i\right) \cup H \cup G$$
$$\phi(H) = \phi((A \cap B)\setminus C) = \phi(A \cap B)\setminus\phi(C)$$
$$= \left(\left(\bigcup_{i=2}^{\infty} C_i\right) \cup \left(\bigcup_{i=2}^{\infty} B_i\right) \cup H \cup G\right)\setminus \left(\left(\bigcup_{i=2}^{\infty} C_i\right) \cup \left(\bigcup_{i=2}^{\infty} B_i\right)\right)$$
$$= H \cup G$$

Then $x(H \cup G) = x(\phi(H)) = x(H)$ and $x(G) = x((H \cup G)\setminus H) = x(H \cup G) \ominus x(H) = 0$.

We denote $E_i = \phi^i(E)$, $i = 1, 2, \ldots$. Evidently $E_i \subseteq C_i \; \forall i = 1, 2, \ldots$, and therefore $E_i \cap E_j = \emptyset$ for every $i \neq j$, $\phi(E_i) = E_{i+1}$, $\forall i = 1, 2, \ldots$. Then

$$\phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=2}^{\infty} E_i, \qquad x\left(\bigcup_{i=2}^{\infty} E_i\right) = x\left(\phi\left(\bigcup_{i=1}^{\infty} E_i\right)\right) = x\left(\bigcup_{i=1}^{\infty} E_i\right)$$

Therefore

$$x(E) = x(\phi(E)) = x(E_1) = x\left(\left(\bigcup_{i=1}^{\infty} E_i\right)\setminus\left(\bigcup_{i=2}^{\infty} E_i\right)\right)$$
$$= x\left(\bigcup_{i=1}^{\infty} E_i\right) \ominus x\left(\bigcup_{i=2}^{\infty} E_i\right) = 0$$

Evidently

$$(A\setminus B)\setminus E = \bigcup_{i=1}^{\infty} (A_{i-1}\setminus A_i), \qquad (B\setminus A)\setminus G = \bigcup_{i=1}^{\infty} B_i$$

$$x((A\setminus B)\setminus E) = x(A\setminus B), \qquad x(B\setminus A)\setminus G) = x(B\setminus A)$$

The mapping ψ: $\mathcal{B}(R)_{(A\setminus B)\setminus E}(R) \to \mathcal{B}(R)_{(B\setminus A)\setminus G}(R)$ defined by

$$\psi(M) = \bigcup_{i=1}^{\infty} \phi^i(M \cap (A_{i-1}\setminus A_i))$$

for every $M \in (A\backslash B)\backslash E$ is an isomorphism and

$$x(\psi(M)) = x\left(\bigcup_{i=1}^{\infty} \phi^i(M \cap (A_{i-1}\backslash A_i))\right) = \bigoplus_{i=1}^{\infty} x(\phi^i(M \cap (A_{i-1}\backslash A_i)))$$

$$= \bigoplus_{i=1}^{\infty} x(M \cap (A_{i-1}\backslash A_i)) = x\left(\bigcup_{i=1}^{\infty} (M \cap (A_{i-1}\backslash A_i))\right) = x(M)$$

Therefore $(A\backslash B)\ \alpha_x\ (B\backslash A)$. ■

The consequence of Proposition 4 is the following assertion.

Theorem 3. Let $A_1, A, B_1, B \in \mathcal{B}(R)$, $A_1\ \alpha_x\ B_1$, $A\ \alpha_x\ B$, and $A_1 \subseteq A$, $B_1 \subseteq B$. Then $(A\backslash A_1)\ \alpha_x\ (B\backslash B_1)$.

Proof. With respect to Corollary 2 we may assume that $A_1 \simeq_x B_1$ and $A \simeq_x B$. Let ϕ, ψ be the isomorphisms $\phi: \mathcal{B}_A(R) \to \mathcal{B}_B(R)$ and $\psi: \mathcal{B}_{A_1}(R) \to \mathcal{B}_{B_1}(R)$. Then $\phi(A_1)\ \alpha_x\ B_1$. Indeed, $(\psi^{-1} \circ \phi)$ is an isomorphism, $(\psi^{-1} \circ \phi): \mathcal{B}_{B_1}(R) \to \mathcal{B}_{\phi(A_1)}(R)$, and

$$x((\psi^{-1} \circ \phi)(E)) = x(\phi(\psi^{-1}(E))) = x(\psi^{-1}(E)) + x(\psi(\psi^{-1}(E))) = x(E)$$

for every $E \in \mathcal{B}_{B_1}(R)$. By Proposition 4, $(\phi(A_1)\backslash B_1)\ \alpha_x\ (B_1\backslash \phi(A_1))$. The mapping $\eta: \mathcal{B}_{A\backslash A_1}(R) \to \mathcal{B}_{B\backslash B_1}(R)$,

$$\eta(E) = \phi(\phi^{-1}(\phi(E) \cap (B\backslash B_1))) \cup \phi(\psi^{-1}(\phi(E) \cap B_1))$$

for every $E \in \mathcal{B}_{A\backslash A_1}(R)$ is an isomorphism and

$$\begin{aligned} x(\eta(E)) &= x(\phi(\phi^{-1}(\phi(E) \cap (B\backslash B_1))) \oplus x(\phi(\psi^{-1}(\phi(E) \cap B_1))) \\ &= x(\phi(E) \cap (B\backslash B_1)) \oplus x(\phi(E) \cap B_1) \\ &= x((\phi(E) \cap (B\backslash B_1)) \cup (\phi(E) \cap B_1)) \\ &= x(\phi(E)) = x(E) \end{aligned}$$

for every $E \in \mathcal{B}_{A\backslash A_1}(R)$. Therefore $(A\backslash A_1)\ \alpha_x\ (B\backslash B_1)$. ■

Theorem 3 enables us to define a partial binary operation $\ominus$ on the factor space $\mathcal{B}(R)/\alpha_x$ in the following way.

Definition 4. Let $[A], [B] \in \mathcal{B}(R)/\alpha_x$. Now, $[B] \ominus [A]$ is defined if and only if $[A] \leq [B]$ (i.e., there exist $A_1 \in [A]$ and $B_1 \in [B]$ such that $A_1 \subseteq B_1$) and $[B] \ominus [A] = [B_1\backslash A_1]$.

Theorem 4. A partial binary operation $\ominus$ is a difference on $\mathcal{B}(R)/\alpha_x$.

Proof. The proof is obvious.

The element $[R]$ is the greatest element in $\mathcal{B}(R)/\alpha_x$ and therefore $(\mathcal{B}(R)/\alpha_x, \leq, \ominus, [R])$ is a D-poset.

Proposition 5. Let $[A]$, $[B]$ be two elements of $\mathcal{B}(R)/\alpha_x$. Let $B_1, B_2 \in [B]$. Then

$$\{[A \cap B_1], A \in [A]\} = \{[A \cap B_2], A \in [A]\}$$

Proof. Let $A_1 \in [A]$ be an arbitrary element. Without loss of generality we can to assume that $B_1 \simeq_x B_2$, and by Proposition 4, $(B_1 \backslash B_2) \simeq_x (B_2 \backslash B_1)$.

Let ψ be an isomorphism, ψ: $\mathcal{B}_{B_1\backslash B_2}(R) \to \mathcal{B}_{B_2\backslash B_1}(R)$, such that $x(\psi(E)) = x(E)$ for every $E \in \mathcal{B}_{B_1\backslash B_2}(R)$. We denote

$$D_1 = (\psi^{-1}(A_1 \cap (B_2\backslash B_1)))\backslash A_1$$

$$D_2 = (A_1 \cap (B_1\backslash B_2))\backslash\psi^{-1}(A_1 \cap (B_2\backslash B_1))$$

$$D_3 = (\psi^{-1}(A_1 \cap (B_2\backslash B_1))) \cap A_1$$

Obviously $A_1 \cap B_1 = A_1 \cap B_1 \cap B_2 \cup D_2 \cup D_3$. If $A = (A_1\backslash(\psi(D_1) \cup D_2)) \cup D_1 \cup \psi(D_2)$, then $A \,\alpha_x\, A_1$ and $A \cap B_2 = A_1 \cap B_1 \cap B_2 \cup \psi(D_2) \cup \psi(D_3)$, and therefore $A_1 \cap B_1 \,\alpha_x\, A \cap B_2$, i.e., $[A_1 \cap B_1] \in \{[A \cap B_2], A \in [A]\}$. We have

$$\{[A \cap B_1], A \in [A]\} \subseteq \{[A \cap B_2], A \in [A]\}$$

Similarly we prove that $\{[A \cap B_2], A \in [A]\} \subseteq \{[A \cap B_1], A \in [A]\}$. Therefore $\{[A \cap B_1], A \in [A]\} = \{[A \cap B_2], A \in [A]\}$. ■

Proposition 6. Let $M \in [A_1 \cap B_1]$, $A_1 \in [A]$, $B_1 \in [B]$. Then there exist $A \in [A]$, $B \in [B]$ such that $M = A \cap B$.

Proof. Let us asume that $M \simeq_x (A_1 \cap B_1)$ and so $M \setminus (A_1 \cap B_1) \simeq_x (A_1 \cap B_1) \setminus M$.

Let ψ be an isomorphism, ψ: $\mathcal{B}_{(A_1\cap B_1)\backslash M}(R) \to \mathcal{B}_{M\backslash(A_1\cap B_1)}(R)$, such that $x(\psi(E)) = x(E)$ for every $E \in \mathcal{B}_{(A_1\cap B_1)\backslash M}(R)$ we put

$$A = (A_1\backslash\psi^{-1}(M\backslash A_1)) \cup (M\backslash A_1), \qquad B = (B_1\backslash\psi^{-1}(M\backslash B_1)) \cup (M\backslash B_1)$$

Then $A \in [A]$, $B \in [B]$, and $A \cap B = M$. ■

Proposition 7. Let $[D] = \max\{[A \cap B], A \in [A], B \in [B]\}$. Then $[D] = [A] \wedge [B]$.

Proof. By Proposition 6, for every $D \in [D]$ there exist $A \in [A]$, $B \in [B]$ such that $D = A \cap B$. Then $D \subseteq A$, $D \subseteq B$ and therefore $[D] \leq [A]$, $[D] \leq [B]$.

Let $[C] \in \mathcal{B}(R)/\alpha_x$, $[C] \leq [A]$, $[C] \leq [B]$, i.e., for every $C \in [C]$ there exist $A_C \in [A]$, $B_C \in [B]$ such that $C \subseteq A_C$, $C \subseteq B_C$. Then $C \subseteq A_C \cap B_C$ since $[C] \leq [D]$. ■

Proposition 8. Let $[A]$, $[B]$ be two elements from $\mathcal{B}(R)/\alpha_x$, $B \in [B]$. Then for every two elements $[C_1], [C_2] \in \{[A \cap B], A \in [A]\}$ there exists an element $[C] \in \{[A \cap B], A \in [A]\}$ such that $[C_1] \leq [C]$ and $[C_2] \leq [C]$.

Proof. Let $A_1, A_2 \in [A]$ be such that $[A_1 \cap B] = [C_1]$, $[A_2 \cap B] = [C_2]$. Without loss of generality we assume that $(A_1 \backslash A_2) \simeq_x (A_2 \backslash A_1)$. Let ϕ be an isomorphism $\phi : \mathcal{B}_{A_1 \backslash A_2}(R) \to \mathcal{B}_{A_2 \backslash A_1}(R)$ such that $x(\phi(E)) = x(E)$ for every $E \in \mathcal{B}_{A_1 \backslash A_2}(R)$. We denote

$$D_2 = [(A_2 \backslash A_1) \backslash \phi((A_1 \backslash A_2) \cap B)] \cap B, \qquad D_1 = \phi^{-1}(\phi((A_1 \backslash A_2) \cap B) \backslash B)$$

and we put $A_1' = (A_1 \backslash \phi^{-1}(D_2)) \cup D_2$, $A_2' = (A_2 \backslash \phi(D_1)) \cup D_1$. Evidently $A_1' \alpha_x A_2'$, $(A_1' \cap B)\ \alpha_x\ (A_2' \cap B)$ and $A_1 \cap B \subseteq A_1' \cap B$, $A_2 \cap B \subseteq A_2' \cap B$. Therefore we put $[C] = [A_1'] = [A_2']$. ■

Theorem 5. Let $x{:}\mathcal{B}(R) \to T$ be an observable on a D-poset T. Let the spectrum of the observable x be finite. Then $\mathcal{B}(R)/\alpha_x$ is a D-lattice of pairwise compatible elements, which is a Boolean D-poset (Chovanec and Kôpka, 1995).

Proof. Let $[A], [B] \in \mathcal{B}(R)/\alpha_x$ be arbitrary two elements. By Proposition 8 and Proposition 7 there exists

$$[D] = \max\{[A \cap B], A \in [A], B \in [B]\} = [A] \wedge [B]$$

Therefore $\mathcal{B}(R)/\alpha_x$ is a D-lattice.

Let $D \in [D]$. Then there exist elements $A \in [A]$ and $B \in [B]$ such that $D = A \cap B$. Then

$$[A] \ominus [D] = [A \backslash D] = [A \backslash B] \leq [R \backslash B] = [R] \ominus [B] = [B]^{\perp}$$

which implies the compatibility of $[A]$ and $[B]$. ■

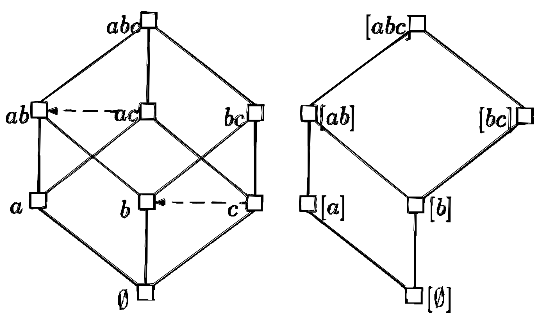

Fig. 1.

Example 1. Let *a, b, c* be different real numbers. Let *x* be the observable on a D-poset of the unit interval [0, 1], defined by.

$$x(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.4 & \text{if } a \in E,\, b,\, c \notin E \\ 0.3 & \text{if } b \in E,\, a,\, c \notin E \\ 0.3 & \text{if } c \in E,\, a,\, b \notin E \\ 0.7 & \text{if } a,\, b \in E,\, c \notin E \\ 0.7 & \text{if } a,\, c \in E,\, b \notin E \\ 0.6 & \text{if } b,\, c \in E,\, a \notin E \\ 1 & \text{if } a,\, b,\, c \in E \end{cases}$$

Then the factorization by Theorem 5 is sketched in Fig. 1.

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